A Machine Learning Approach to Optimal Control

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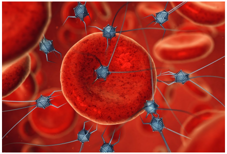
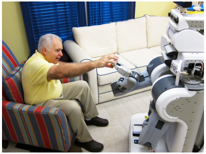
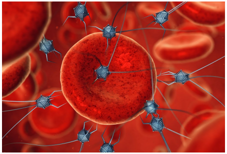
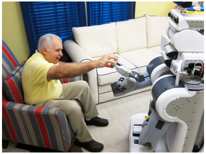
Autonomous Robots

- Vision: Autonomous robots support humans in everyday activities ▶▶ Fast learning and automatic adaptation

Autonomous Robots

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- Currently: Data-hungry learning or human guidance

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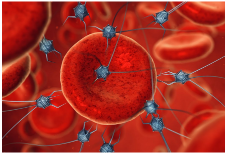

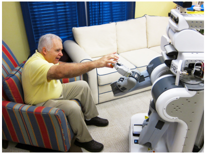


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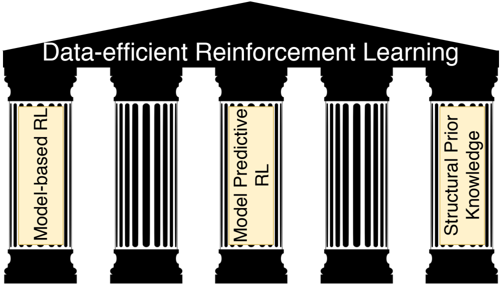
Fully **autonomous learning and decision making with little data** in real-life situations

Central Problem

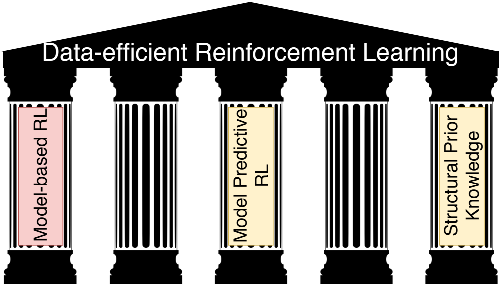
Data-Efficient Reinforcement Learning

Ability to learn and make decisions in complex domains without requiring large quantities of data

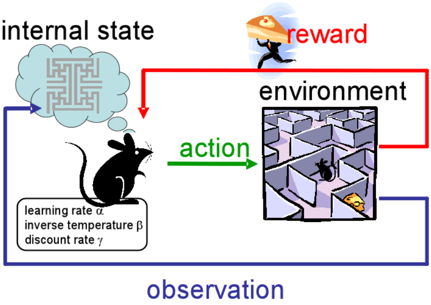
Data-Efficient RL for Autonomous Robots



1. **Model-based RL**
   - Data-efficient decision making
2. **Model predictive RL**
   - Speed up learning further by online planning
3. **Incorporation of structural prior knowledge**
   - Exploit physical and geometric properties to constrain the learning problem


Data-efficient Reinforcement Learning
Model-based RL
Model Predictive RL
Structural Prior Knowledge

Reinforcement Learning



- Learn to solve a task
- Trial-and-error interaction with the environment
- Feedback via reward/cost function

Reinforcement Learning and Optimal Control

$$\boldsymbol{x}_{t+1} = f(\boldsymbol{x}_t, \boldsymbol{u}_t) + \boldsymbol{w}\,, \qquad \boldsymbol{u}_t = \pi(\boldsymbol{x}_t, \boldsymbol{\theta})$$

State Control Policy Policy parameters

Transition function

$$\boldsymbol{x}_{t+1} = {\color{red}f}(\boldsymbol{x}_t, \boldsymbol{u}_t) + \boldsymbol{w}\,, \qquad \boldsymbol{u}_t = \pi(\boldsymbol{x}_t, \boldsymbol{\theta})$$

State

Transition function

Control

Policy

Policy parameters

Objective (Controller Learning)

Find policy parameters $\boldsymbol{\theta}^*$ that minimize the expected long-term cost

$$J(\boldsymbol{\theta}) = \sum\nolimits_{t=1}^{T} \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]\,, \qquad p(\boldsymbol{x}_0) = \mathcal{N}\big(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\big)\,.$$

Instantaneous cost $c(\boldsymbol{x}_t)$, e.g., $\|\boldsymbol{x}_t - \boldsymbol{x}_{\text{target}}\|^2$

- Typical objective in optimal control and reinforcement learning (Bertsekas, 2005; Sutton & Barto, 1998)

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

PILCO Framework: High-Level Steps

1. Probabilistic model for transition function f
   ▶▶ System identification

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Fast Reinforcement Learning

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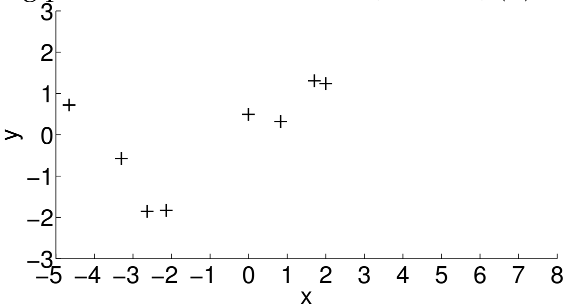
Model Learning (System Identification)

Model learning problem: Find a function $f : x \mapsto f(x) = y$



Observed function values

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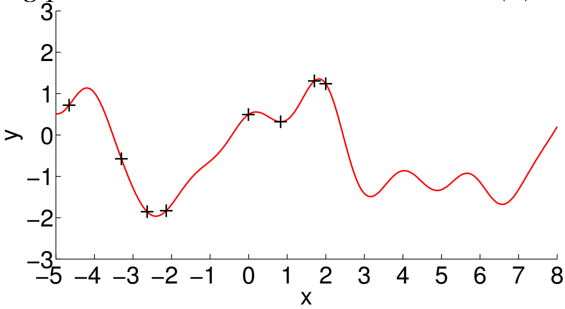

Plausible model

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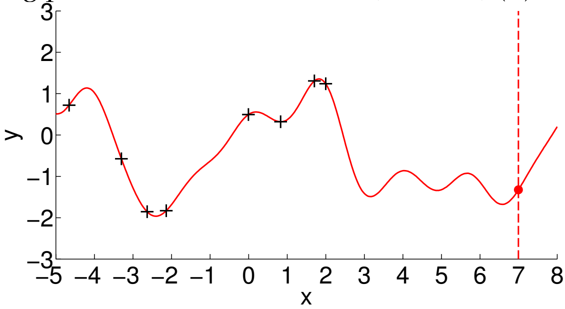

Plausible model

Predictions? Decision Making?

Model Learning (System Identification)

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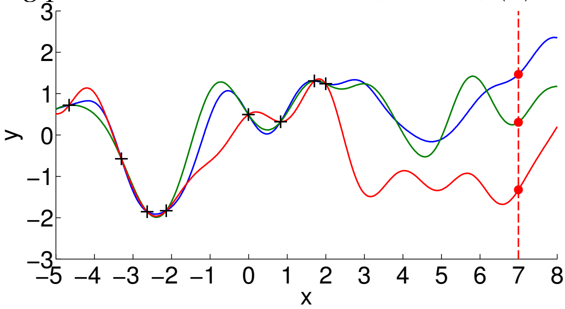
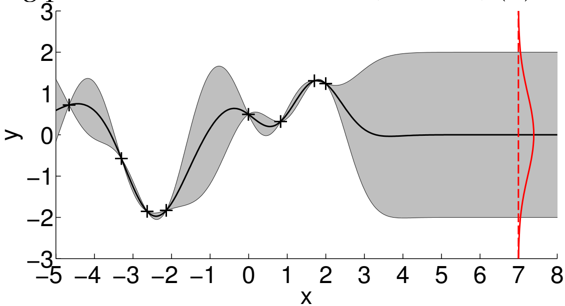


More plausible models

Predictions? Decision Making? Model Errors!

Model Learning (System Identification)

Model learning problem: Find a function $f : x \mapsto f(x) = y$



Distribution over plausible functions

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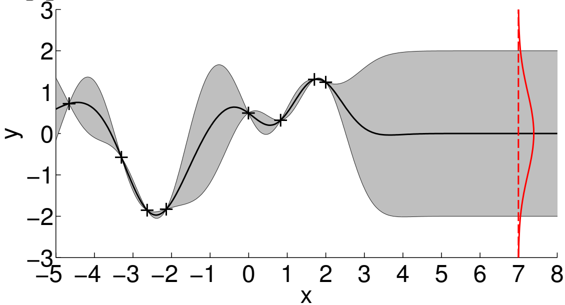

Distribution over plausible functions

- Express uncertainty about the underlying function to be robust to model errors
- Gaussian process for model learning (Rasmussen & Williams, 2006)

Introduction to Gaussian Processes

- Flexible Bayesian regression method
- Probability distribution over functions
- Fully specified by
  - Mean function m (average function)
  - Covariance function k (assumptions on structure)

$$k(\boldsymbol{x}_p, \boldsymbol{x}_q) = \mathrm{Cov}[f(\boldsymbol{x}_p), f(\boldsymbol{x}_q)]$$

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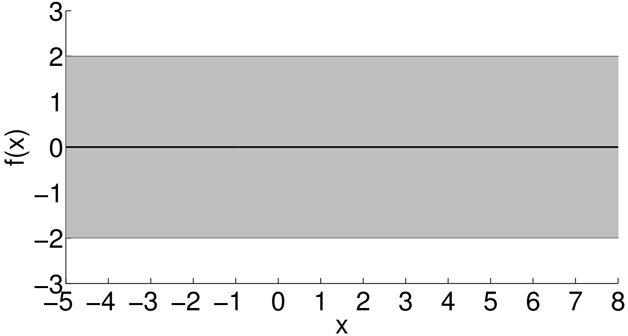
$$k(\boldsymbol{x}_p, \boldsymbol{x}_q) = \mathrm{Cov}[f(\boldsymbol{x}_p), f(\boldsymbol{x}_q)]$$

- Posterior predictive distribution at $\boldsymbol{x}_*$ is Gaussian (Bayes' theorem):

$$p(f(\boldsymbol{x}_*)|\boldsymbol{x}_*, \boldsymbol{X}, \boldsymbol{y}) = \mathcal{N}\big(f(\boldsymbol{x}_*)\,|\,m(\boldsymbol{x}_*),\, \sigma^2(\boldsymbol{x}_*)\big)$$

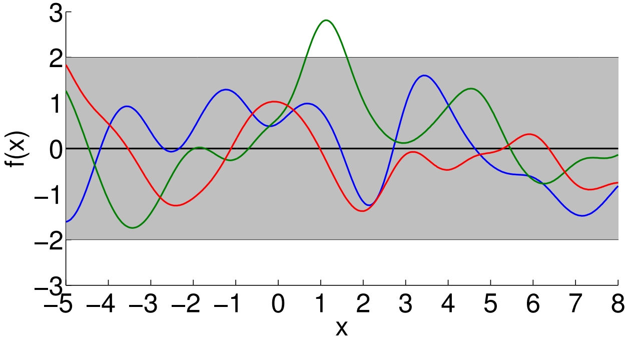
Test input Training data



Prior belief about the function

Predictive (marginal) mean and variance:

$$\mathbb{E}[f(\boldsymbol{x}_*)|\boldsymbol{x}_*, \varnothing] = m(\boldsymbol{x}_*) = 0$$
$$\mathbb{V}[f(\boldsymbol{x}_*)|\boldsymbol{x}_*, \varnothing] = \sigma^2(\boldsymbol{x}_*) = k(\boldsymbol{x}_*, \boldsymbol{x}_*)$$



Prior belief about the function

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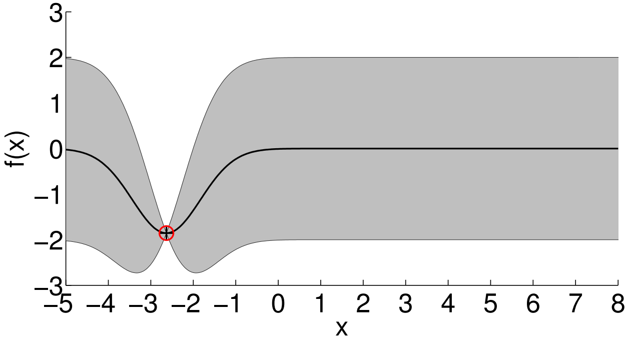

Posterior belief about the function

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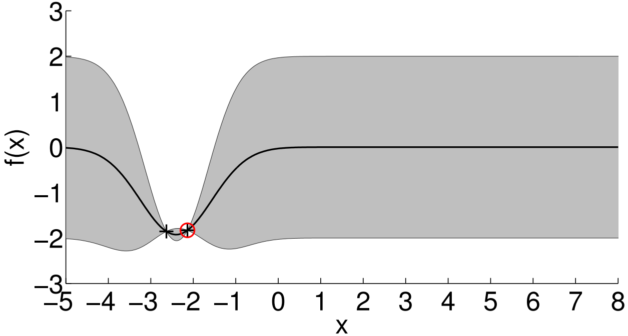

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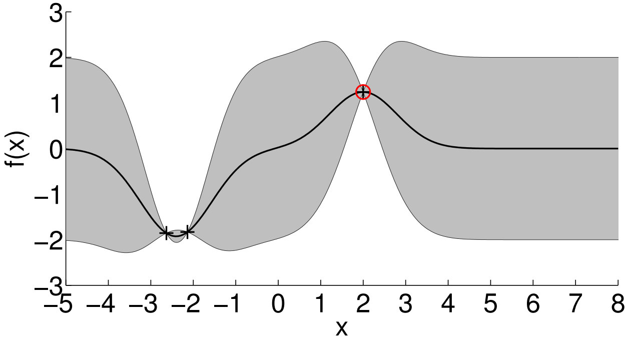

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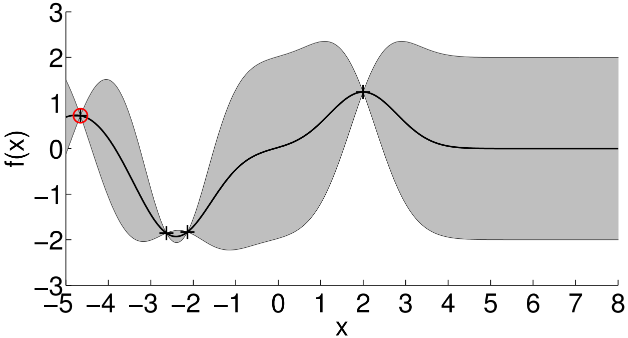

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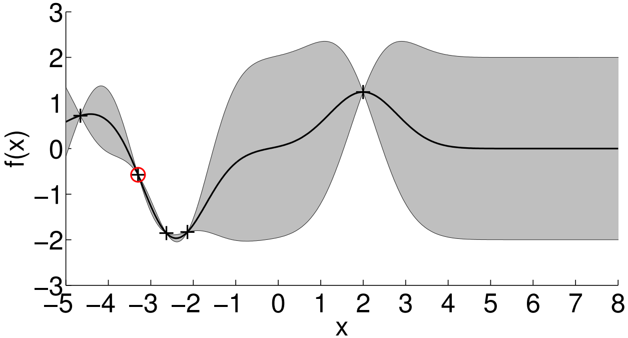

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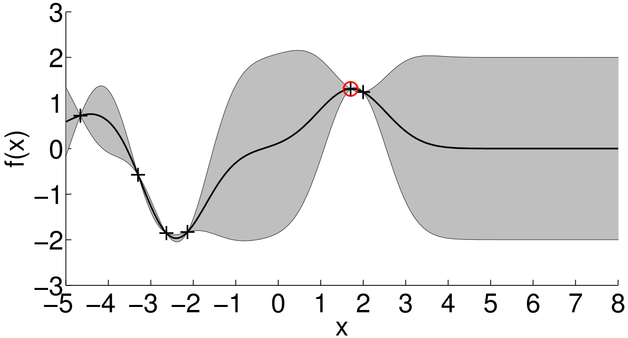
Intuitive Introduction to Gaussian Processes



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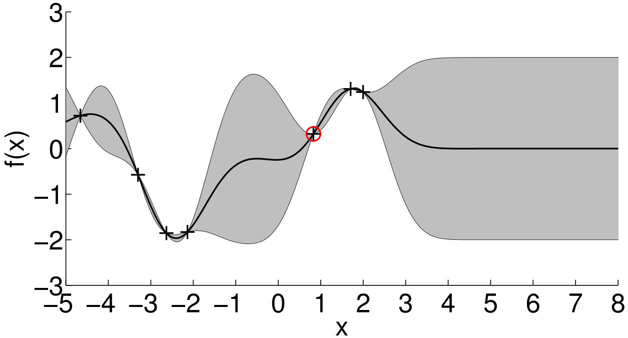

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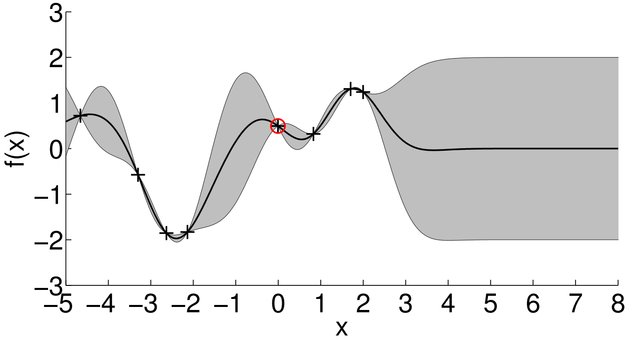

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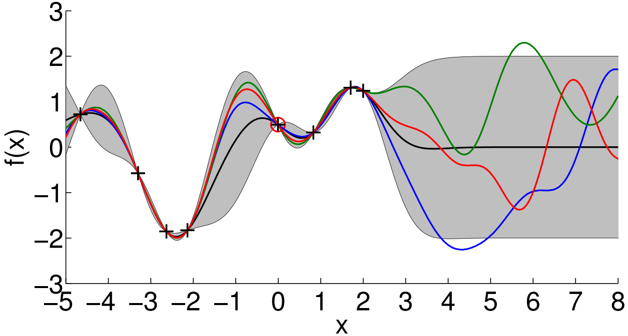

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Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

PILCO Framework: High-Level Steps

1. Probabilistic model for transition function f
   ▶▶ System identification
2. **Compute long-term predictions** $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \dots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
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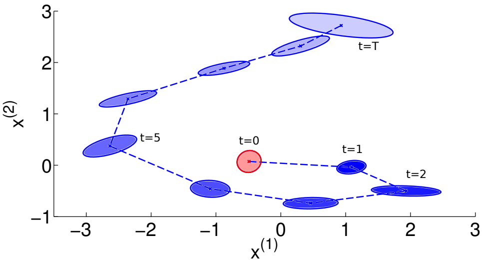
Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Long-Term Predictions



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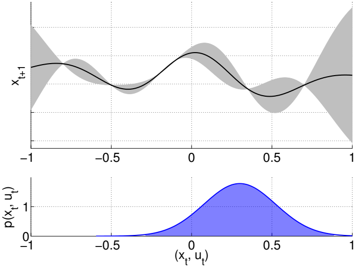

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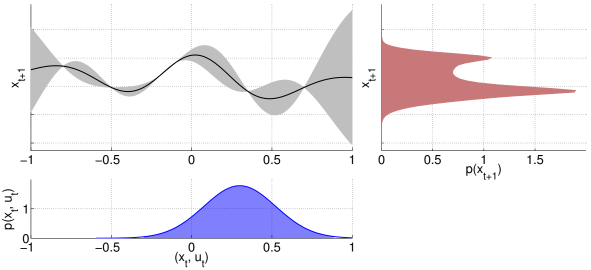

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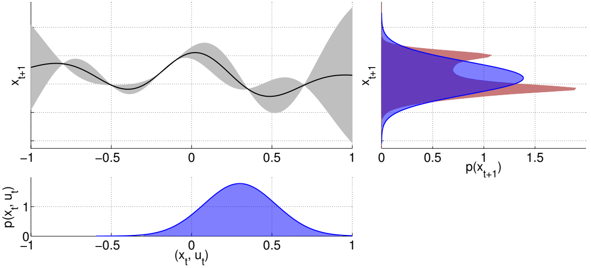
Long-Term Predictions



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▶▶ GP moment matching
(Girard et al., 2002; Quiñonero-Candela et al., 2003)

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2. Compute long-term predictions $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
3. **Policy improvement**
   - Compute expected long-term cost $J(\boldsymbol{\theta})$
   - Find parameters $\boldsymbol{\theta}$ that minimize $J(\boldsymbol{\theta})$
4. Apply controller

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Policy Improvement

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

- Know how to predict $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

- Know how to predict $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \dots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
- Compute

$$\mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}] = \int c(\boldsymbol{x}_t)\mathcal{N}\big(\boldsymbol{x}_t \,|\, \boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\big) d\boldsymbol{x}_t\,, \quad t = 1, \dots, T\,,$$

and sum them up to obtain $J(\boldsymbol{\theta})$

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Policy Improvement

Objective

Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

- Know how to predict $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \dots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
- Compute

$$\mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}] = \int c(\boldsymbol{x}_t)\mathcal{N}\big(\boldsymbol{x}_t \,|\, \boldsymbol{\mu}_t,\, \boldsymbol{\Sigma}_t\big)d\boldsymbol{x}_t\,, \quad t = 1, \dots, T\,,$$

  and sum them up to obtain $J(\boldsymbol{\theta})$
- Analytically compute gradient $\mathrm{d}J(\boldsymbol{\theta})/\mathrm{d}\boldsymbol{\theta}$
- Standard gradient-based optimizer (e.g., BFGS) to find $\boldsymbol{\theta}^*$

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

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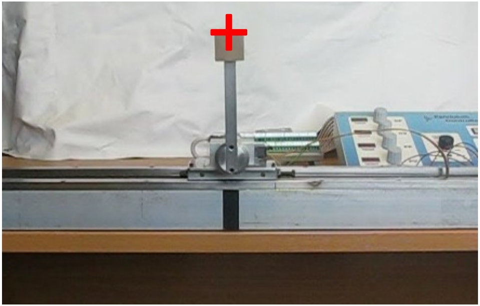
Minimize expected long-term cost $J(\boldsymbol{\theta}) = \sum_t \mathbb{E}[c(\boldsymbol{x}_t)|\boldsymbol{\theta}]$

PILCO Framework: High-Level Steps

1. Probabilistic model for transition function f
   ▸▸ System identification
2. Compute long-term predictions $p(\boldsymbol{x}_1|\boldsymbol{\theta}), \ldots, p(\boldsymbol{x}_T|\boldsymbol{\theta})$
3. Policy improvement
4. **Apply controller**

Deisenroth et al. (IEEE-TPAMI, 2015): *Gaussian Processes for Data-Efficient Learning in Robotics and Control*

Standard Benchmark: Cart-Pole Swing-up



- Swing up and balance a freely swinging pendulum on a cart
- No knowledge about nonlinear dynamics ⏩ Learn from scratch
- Cost function $c(\boldsymbol{x}) = 1 - \exp(-\|\boldsymbol{x} - \boldsymbol{x}_{\text{target}}\|^2)$

- Code: https://github.com/ICL-SML/pilco-matlab

Deisenroth & Rasmussen (ICML, 2011): *PILCO: A Model-based and Data-efficient Approach to Policy Search*

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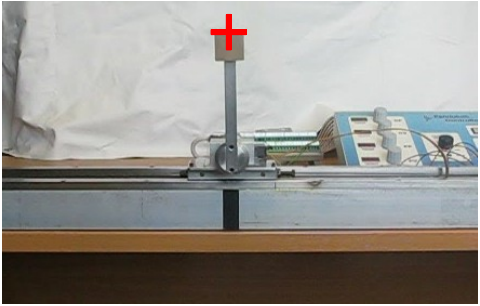

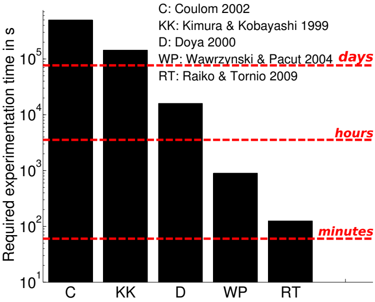

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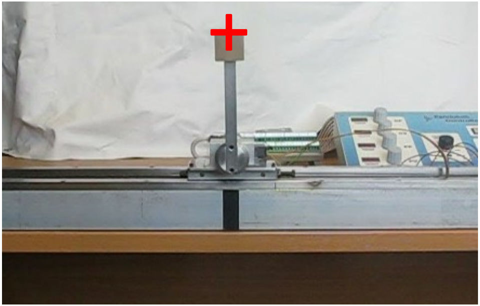
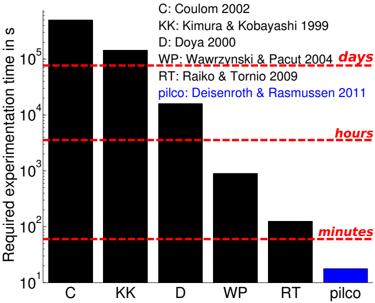
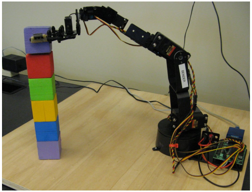




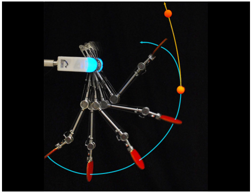
- Swing up and balance a freely swinging pendulum on a cart
- No knowledge about nonlinear dynamics ⏩ Learn from scratch
- Cost function $c(\boldsymbol{x}) = 1 - \exp(-\|\boldsymbol{x} - \boldsymbol{x}_{\text{target}}\|^2)$
- **Unprecedented learning speed** compared to state-of-the-art
- Code: `https://github.com/ICL-SML/pilco-matlab`

Deisenroth & Rasmussen (ICML, 2011): *PILCO: A Model-based and Data-efficient Approach to Policy Search*

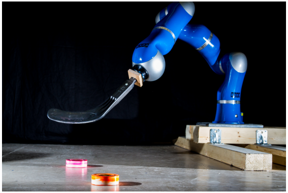
Wide Applicability



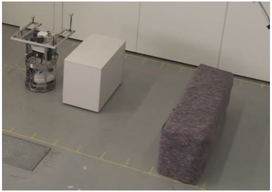
with D Fox



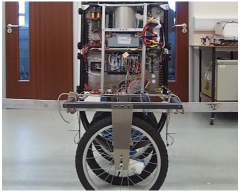
with P Englert, A Paraschos, J Peters



with A Kupcsik, J Peters, G Neumann



B Bischoff (Bosch), ESANN 2013



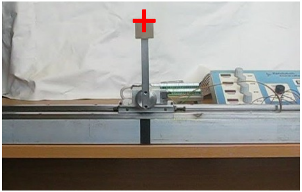
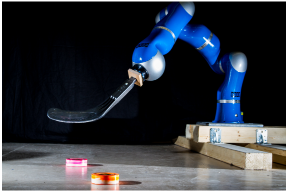
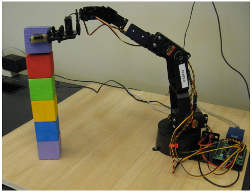
A McHutchon (U Cambridge)

B Bischoff (Bosch), ECML 2013

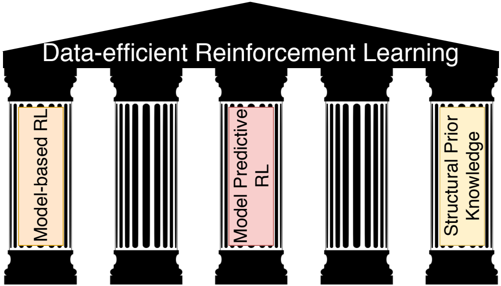
- Application to a wide range of robotic systems

Deisenroth et al. (RSS, 2011): *Learning to Control a Low-Cost Manipulator using Data-efficient Reinforcement Learning*
Englert et al. (ICRA, 2013): *Model-based Imitation Learning by Probabilistic Trajectory Matching*
Deisenroth et al. (ICRA, 2014): *Multi-Task Policy Search for Robotics*
Kupcsik et al. (AIJ, 2017): *Model-based Contextual Policy Search for Data-Efficient Generalization of Robot Skills*

Summary (1)

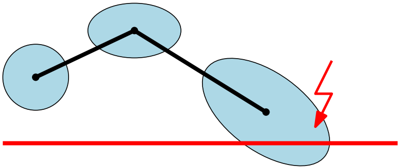




- In robotics, data-efficient learning is critical
- Probabilistic, model-based RL approach
  - Reduce model bias
  - Unprecedented learning speed
  - Wide applicability



Sanket Kamthe

Safe Exploration



- Deal with real-world safety constraints (states/controls)
- Use probabilistic model to predict whether state constraints are violated (e.g., Sui et al., 2015; Berkenkamp et al., 2017)
- Adjust policy if necessary (during policy learning)

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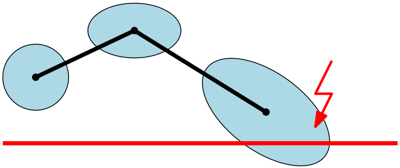

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▶▶ Safe exploration within an MPC-based RL setting

▶▶ Optimize control signals $\boldsymbol{u}_t$ directly (no policy parameters)

Approach

- Idea: Optimize control signals directly (instead of policy parameters)
- Few parameters to optimize ⏩ Low-dimensional search space
- Open-loop control
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- Model predictive control (MPC) turns this into a closed-loop control approach
- Use this within a trial-and-error RL setting

Probabilistic MPC in RL

- Learned GP model for transition dynamics
- Repeat (while executing the policy):
  1. In current state $\boldsymbol{x}_t$, determine optimal control sequence $\boldsymbol{u}_0^*, \ldots, \boldsymbol{u}_{H-1}^*$
  2. Apply first control $\boldsymbol{u}_0^*$ in state $\boldsymbol{x}_t$
  3. Transition to next state $\boldsymbol{x}_{t+1}$
  4. Update GP transition model

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Theoretical Results

- Uncertainty propagation is deterministic (GP moment matching)
  - ⏩ Re-formulate system dynamics:

$$\boldsymbol{z}_{t+1} = f_{MM}(\boldsymbol{z}_t, \boldsymbol{u}_t)$$
$$\boldsymbol{z}_t = \{\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\}$$

⏩ Collects moments

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

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⏩ Collects moments

- Deterministic system function that propagates moments
- Lipschitz continuity (under mild assumptions) implies that we can apply Pontryagin's Minimum Principle
  - Control Hamiltonian $H(\boldsymbol{\lambda}_{t+1}, \boldsymbol{z}_t, \boldsymbol{u}_t)$
  - Adjoint recursion for $\boldsymbol{\lambda}_t$
  - Necessary optimality condition: $\partial H / \partial \boldsymbol{u}_t = \boldsymbol{0}$

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Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

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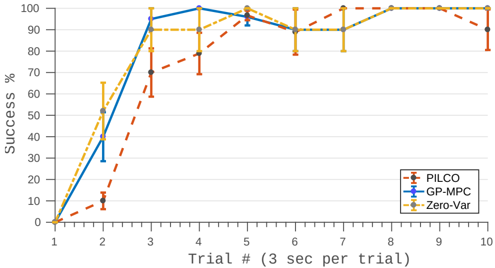
  ⏩ Principled treatment of constraints on controls
- Use predictive uncertainty to check violation of state constraints

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

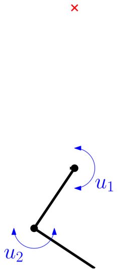
Learning Speed (Cart Pole)

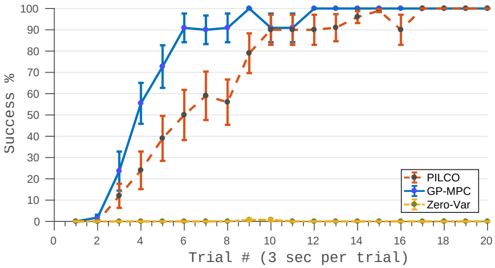


- Zero-Var: Only use the mean of the GP, discard variances for long-term predictions
- MPC: Increased data efficiency (40% less experience required than PILCO)
  ▶▶ MPC more robust to model inaccuracies than a parametrized feedback controller

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Learning Speed (Double Pendulum)





- GP-MPC maintains the same improvement in data efficiency
- Zero-Var fails:
  - Gets stuck in local optimum near start state
  - Insufficient exploration due to lack of uncertainty propagation

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

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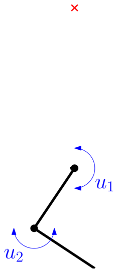
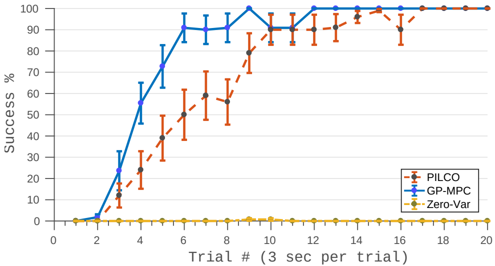
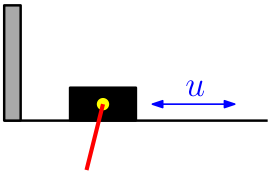


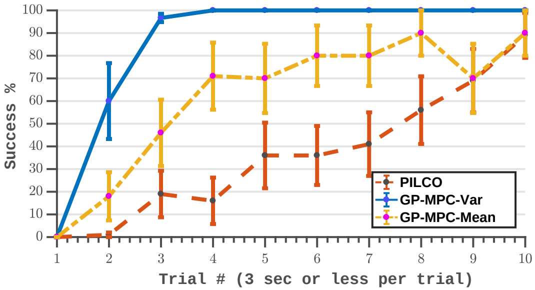


- GP-MPC maintains the same improvement in data efficiency
- Zero-Var fails:
  - Gets stuck in local optimum near start state
  - Insufficient exploration due to lack of uncertainty propagation
- Although MPC is fairly robust to model inaccuracies we cannot get away without uncertainty propagation

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Safety Constraints (Cart Pole)



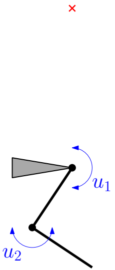


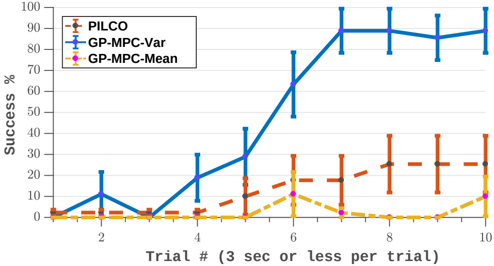
PILCO	16/100	constraint violations
GP-MPC-Mean	21/100	constraint violations
GP-MPC-Var	3/100	constraint violations

Propagating model uncertainty important for safety

Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Safety Constraints (Double Pendulum)

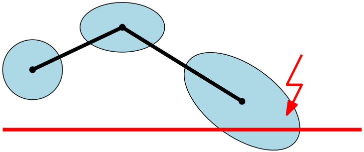


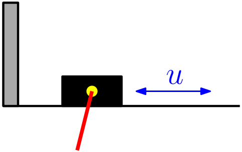


Experiment	Double Pendulum
PILCO	23/100
GP-MPC-Mean	26/100
GP-MPC-Var	11/100

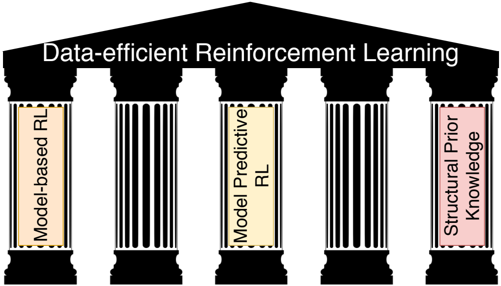
Kamthe & Deisenroth (AISTATS, 2018): *Data-Efficient Reinforcement Learning with Probabilistic Model Predictive Control*

Summary (2)





- Probabilistic prediction models for safe exploration
- Uncertainty propagation can be used to reduce violation of safety constraints
- MPC framework increases robustness to model errors
  ▶▶ Increased data efficiency



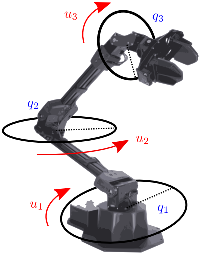
Steindór Sæmundsson

Alexander Terenin

Katja Hofmann

Structural Priors

High-level prior knowledge: e.g., laws of physics or configuration constraints



Equations of motion

$$u = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

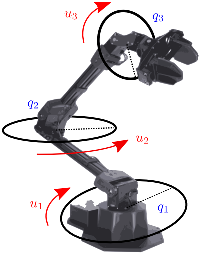
➡ Improve data efficiency and generalization

Variational Integrator Networks (VINs)

Network architectures with built-in physics and geometric structure

Outline:

- How we talk about physics
- How we think about neural networks
- How to encode physics and geometry into architecture



Equations of motion

$$u = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

- General framework:
  classical mechanics, quantum mechanics, relativity
- Global properties:
  conservation laws, configuration manifold, etc.
- Solve differential equations

Physics: Key Objects

- Configuration space:

$$q \in \mathcal{Q}$$

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- Lagrangian (specifies physics):

$$L(q(t), \dot{q}(t)) = K - U = \text{kinetic energy} - \text{potential energy}$$

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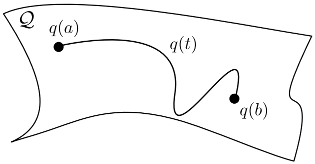
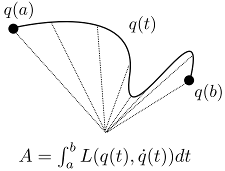
- Lagrangian (specifies physics):

$$L(q(t), \dot{q}(t)) = K - U = \text{kinetic energy} - \text{potential energy}$$

- Action (maps trajectories to real numbers)

$$A = \int_a^b L(q(t), \dot{q}(t))dt$$





Physics: Hamilton's Principle

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Physical paths are stationary points of the action.

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Equations of motion (Euler-Lagrange equation):

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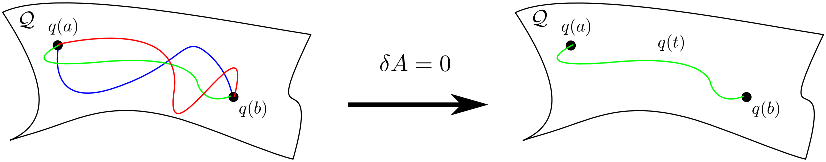
Hamilton's Principle

Physical paths are stationary points of the action.

Equations of motion (Euler-Lagrange equation):

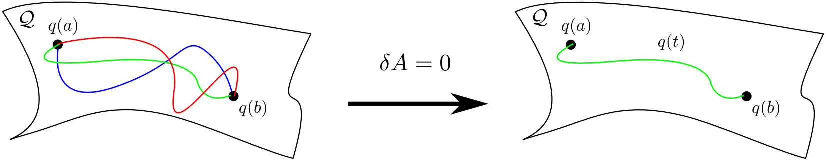
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

The solution $q(t)$ evolves according to the laws of physics.



Physics: Recap

- Lagrangian → Specifies the physics
- Hamilton's principle → Equations of motion
- Solution → Physical path



Neural ODE Perspective

- Residual networks = Learnable approximate ODE solvers

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), t, \theta) \quad \longleftrightarrow \quad \boldsymbol{x}_{t+1} = \boldsymbol{x}_t + f(\boldsymbol{x}(t), \theta)$$

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- **Intuition:** Physical networks = Learnable approximations to equations of motion

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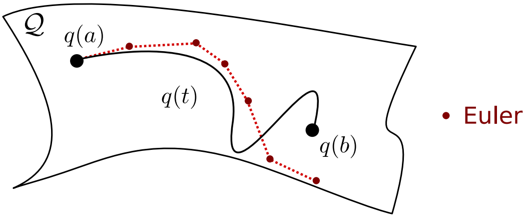
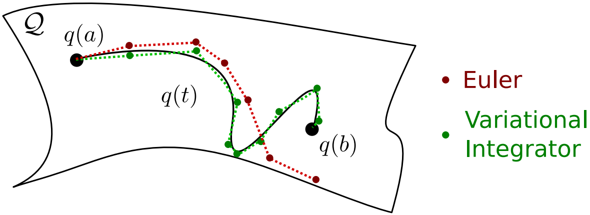
- **Intuition:** Physical networks = Learnable approximations to equations of motion
- **Problem:** Euler discretization leads to significant errors and physically implausible behavior



Variational Integrators

Geometric integrators that preserve global (physical) properties



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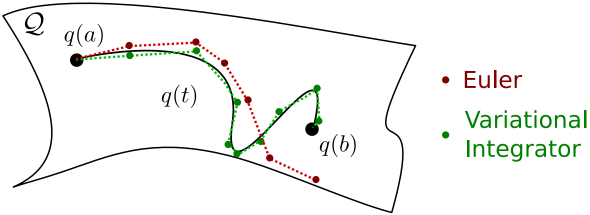


Properties:

- Symplectic (volume preserving)
- Momentum preserving
- Bounded energy behavior

Recipe for Variational Integrator Network

1. Write down parameterized Lagrangian:

$$L_\theta(q(t), \dot{q}(t))$$

Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

Recipe for Variational Integrator Network

1. Write down parameterized Lagrangian:

$$L_\theta(q(t), \dot{q}(t))$$

2. Derive **explicit** variational integrator:

$$\text{Lagrangian:} \quad q_{t+1} = f_\theta(q_t, q_{t-1})$$
$$\text{Hamiltonian:} \quad [q_{t+1}, \dot{q}_{t+1}] = f_\theta(q_t, \dot{q}_t)$$

Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

Recipe for Variational Integrator Network

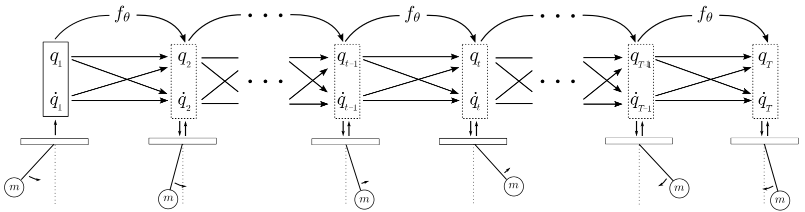
1. Write down parameterized Lagrangian:

$$L_\theta(q(t), \dot{q}(t))$$

2. Derive **explicit** variational integrator:

$$\text{Lagrangian:} \quad q_{t+1} = f_\theta(q_t, q_{t-1})$$
$$\text{Hamiltonian:} \quad [q_{t+1}, \dot{q}_{t+1}] = f_\theta(q_t, \dot{q}_t)$$

3. f_θ defines the network architecture



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VIN Examples

Newtonian Potential System:

$$L_\theta(q(t), \dot{q}(t)) = K_\theta(\dot{q}(t)) - U_\theta(q(t))$$

- Newtonian network on $\mathbb{R}^D$

$$q_{t+1} = 2q_t - q_{t-1} - h^2 f_\theta(q_t)$$

Newtonian Potential System:

$$L_\theta(q(t), \dot{q}(t)) = K_\theta(\dot{q}(t)) - U_\theta(q(t))$$

- Newtonian network on $\mathbb{R}^D$

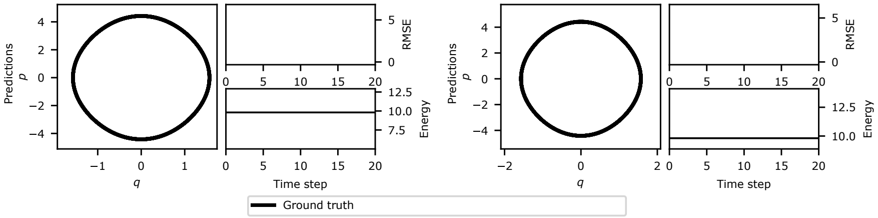
$$q_{t+1} = 2q_t - q_{t-1} - h^2 f_\theta(q_t)$$

- Newtonian network on $SO(2)$

$$\begin{aligned} \sin \Delta q_t &= \sin \Delta q_{t-1} + h^2 r_\theta(q_t) \\ q_{t+1} &= q_t + \Delta q_t \end{aligned}$$

▶▶ Allows us to define dynamics on a manifold

Learning from Noisy Data: Pendulum



Pendulum System. **Left:** 150 observations; **Right:** 750 observations.

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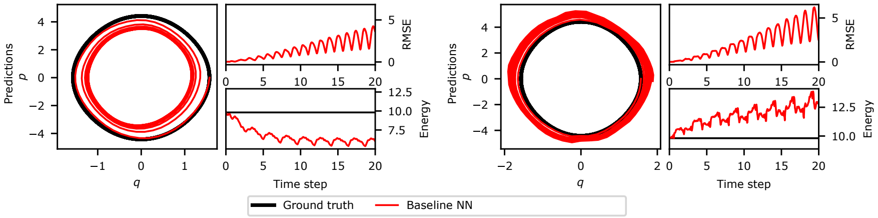
Learning from Noisy Data: Pendulum



Pendulum System. **Left:** 150 observations; **Right:** 750 observations.

- Baseline neural network: Dissipates/adds energy for low and moderate data

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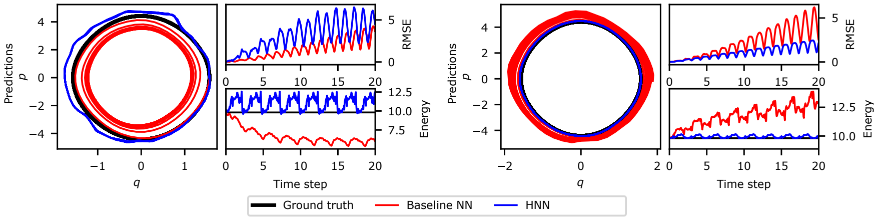
Learning from Noisy Data: Pendulum



Pendulum System. **Left:** 150 observations; **Right:** 750 observations.

- Baseline neural network: Dissipates/adds energy for low and moderate data
- Hamiltonian neural network (Greydanus et al., 2019): Overfits in low-data regime

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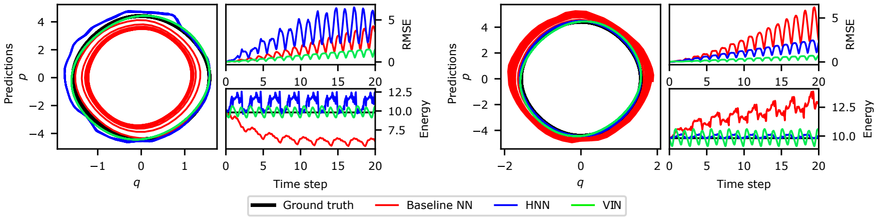
Learning from Noisy Data: Pendulum

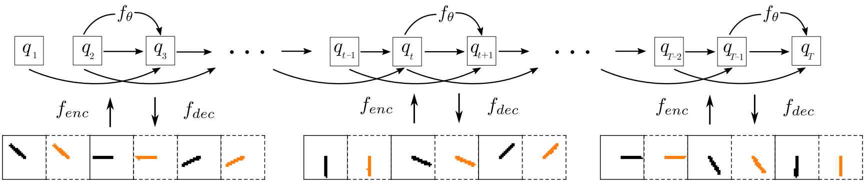


Pendulum System. **Left:** 150 observations; **Right:** 750 observations.

- Baseline neural network: Dissipates/adds energy for low and moderate data
- Hamiltonian neural network (Greydanus et al., 2019): Overfits in low-data regime
- Variational integrator network: Conserves energy and generalizes better in both regimes

Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

Learning from Pixel Data

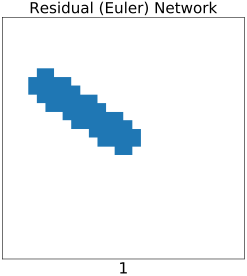


- VIN within variational auto-encoder (VAE) setup:
  - Learn physical system in lower-dimensional latent space
  - Use VIN for long-term forecasting

▶▶ Exploit geometry of the problem for system identification and forecasting

Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

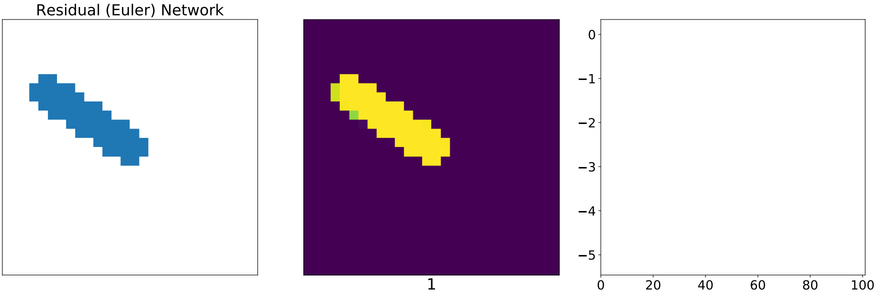
Learning from Pixel Data: Forecasting



- Observations: 28×28 pixel images of pendulum
- Training data: 40 images

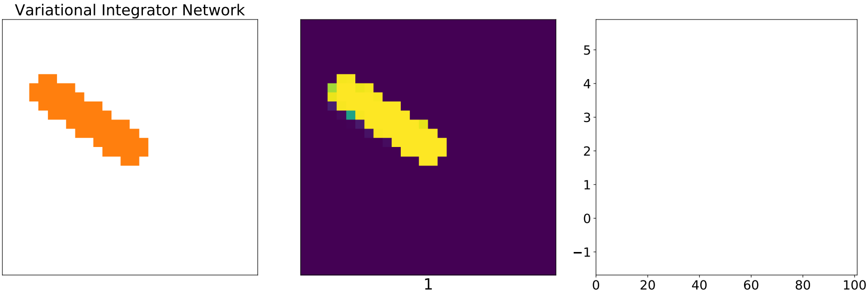
Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

Learning from Pixel Data: Forecasting



- Observations: 28×28 pixel images of pendulum
- Training data: 40 images
- Dynamic VAE: Forecasting is not meaningful

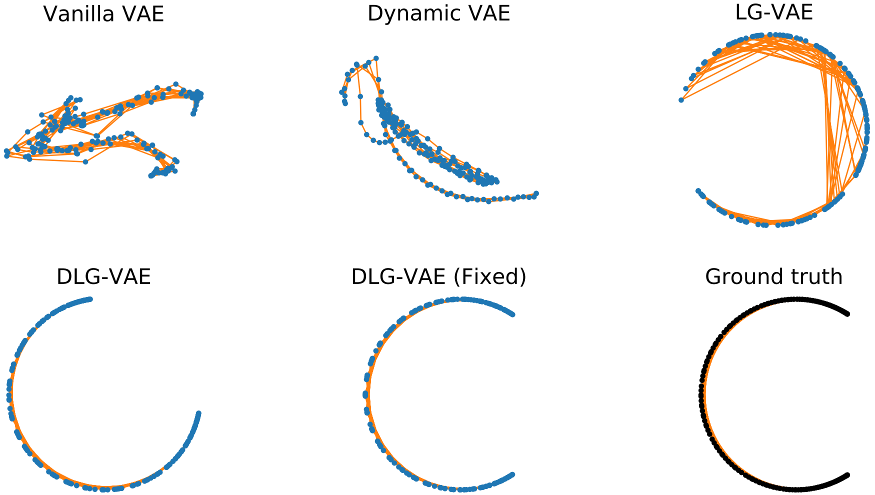
Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*



- Observations: 28×28 pixel images of pendulum
- Training data: 40 images
- Dynamic VAE: Forecasting is not meaningful
- DLG-VAE: Physically meaningful long-term forecasts in latent and observation space

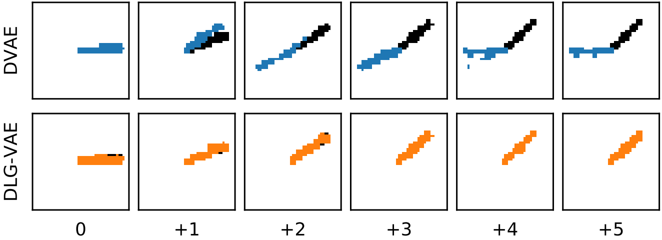
Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

Learning from Pixel Data: Latent Embeddings



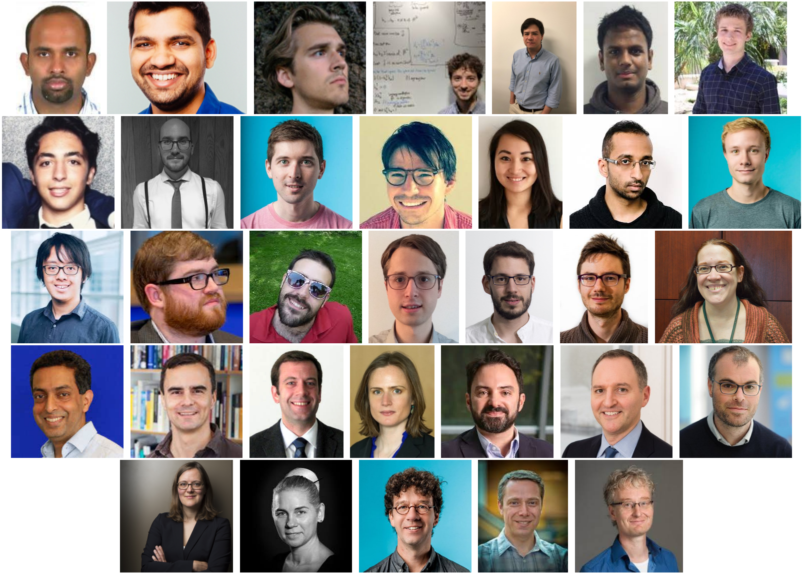
Sæmundsson et al. (arXiv:1910.09349): *Variational Integrator Networks for Physically Meaningful Embeddings*

Summary (3)

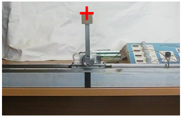


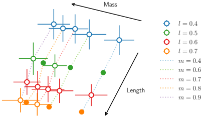
- Variational integrator networks to encode physics and geometric structure ⏩ Interpretability
- Data-efficient learning and physically meaningful long-term forecasts

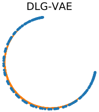
Team and Collaborators



Wrap-up

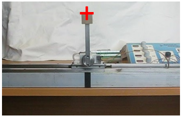


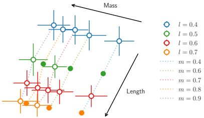


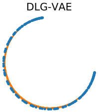


- **Data efficiency** is a practical challenge for autonomous robots
- Three pillars of data-efficient machine learning
  1. Model-based reinforcement learning with learned probabilistic models for fast learning from scratch
  2. Model predictive RL for safe exploration and more robust models
  3. Incorporation of structural priors for learning physically meaningful predictive models

Wrap-up







- **Data efficiency** is a practical challenge for autonomous robots
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ありがとうございました

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GP Moment Matching: Some Details

$$f \sim GP(0, k)\,, \quad \text{Training data: } \boldsymbol{X}, \boldsymbol{y}$$
$$\boldsymbol{x}_* \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Compute $\mathbb{E}[f(\boldsymbol{x}_*)]$

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&= \mathbb{E}_{\boldsymbol{x}_*}\big[\,k(\boldsymbol{x}_*, \boldsymbol{X})(\boldsymbol{K} + \sigma_n^2 \boldsymbol{I})^{-1}\boldsymbol{y}\,\big] \\
&= \boldsymbol{\beta}^\top \int k(\boldsymbol{X}, \boldsymbol{x}_*)\mathcal{N}\big(\boldsymbol{x}_* \,|\, \boldsymbol{\mu},\, \boldsymbol{\Sigma}\big) d\boldsymbol{x}_*
\end{aligned}$$

$\boldsymbol{\beta} := (\boldsymbol{K} + \sigma_n^2 \boldsymbol{I})^{-1}\boldsymbol{y}$ ⏩ independent of $\boldsymbol{x}_*$

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- If k is a Gaussian (squared exponential) kernel, this integral can be solved analytically
- Variance of $f(\boldsymbol{x}_*)$ can be computed similarly