Integration

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Integration problems

- Moment computation

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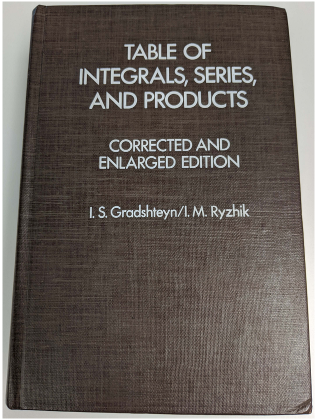
- Planning

$$J^\pi(\boldsymbol{x}(0)) = \int_0^T r(\boldsymbol{x}(t), \boldsymbol{u}(t))|\boldsymbol{x}(0) dt$$

Exact integration



- Compute integrals analytically, if possible (Gradshteyn & Ryzhik, 2007)

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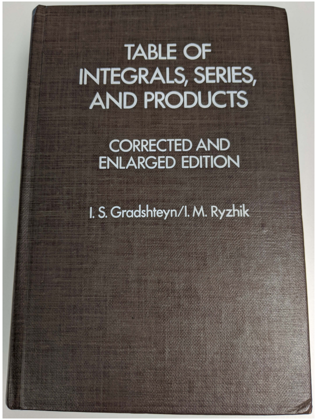

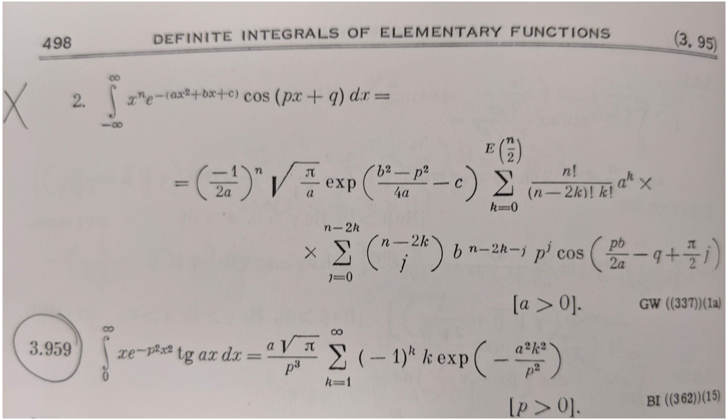

498 DEFINITE INTEGRALS OF ELEMENTARY FUNCTIONS (3.95)

2. $$\int_{-\infty}^{\infty} x^n e^{-(ax^2+bx+c)} \cos(px+q)\,dx =$$

$$= \left(\frac{-1}{2a}\right)^n \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2-p^2}{4a}-c\right) \sum_{k=0}^{E\left(\frac{n}{2}\right)} \frac{n!}{(n-2k)!\,k!} a^k \times$$

$$\times \sum_{j=0}^{n-2k} \binom{n-2k}{j} b^{n-2k-j} p^j \cos\left(\frac{pb}{2a}-q+\frac{\pi}{2}j\right)$$

$[a > 0]$. GW ((337))(1a)

3.959 $$\int_0^\infty x e^{-p^2x^2} \operatorname{tg} ax\,dx = \frac{a\sqrt{\pi}}{p^3} \sum_{k=1}^{\infty} (-1)^k k \exp\left(-\frac{a^2k^2}{p^2}\right)$$

$[p > 0]$. BI ((362))(15)

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Approximate integration

Topic	Useful reference	Video
Numerical integration	Stoer & Bulirsch (2002)	Chapter 1
Bayesian quadrature	Rasmussen & Ghahramani (2003), Gunter et al. (2014)	Chapter 1
Monte-Carlo integration	MacKay (2003), Murray (2015)	Chapter 2
Normalizing flows	Weng (2018), Papamakarios et al. (2019), Kobyzev et al. (2020)	Chapter 3
Inference in time series	Julier & Uhlmann (2004), Särkkä (2013)	Chapter 4

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